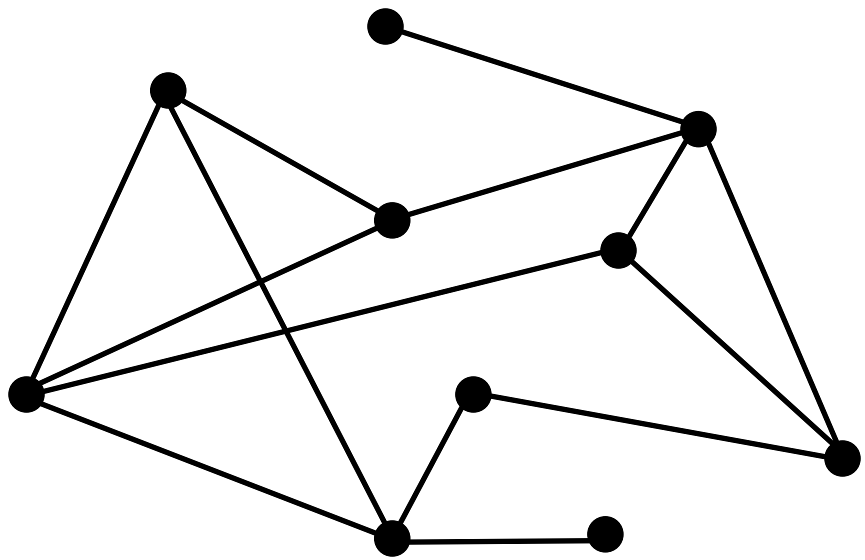
Coloring intersection points of line segments

Sam Lowery
Department of Mathematics & Statistics

2022 Symposium for Student Research, Scholarship, and Creative Activity

Graphs



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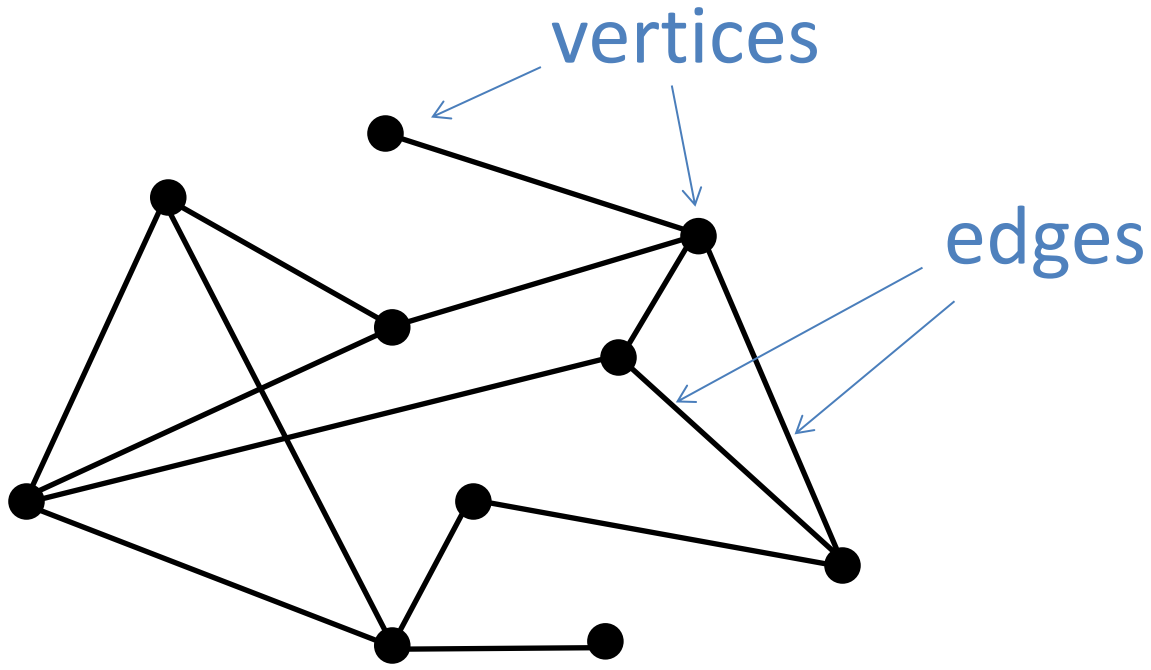
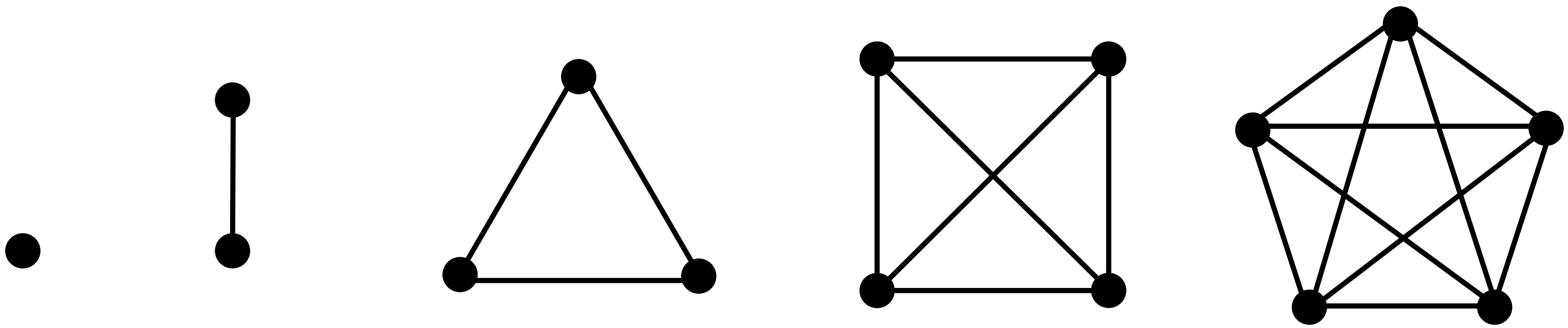


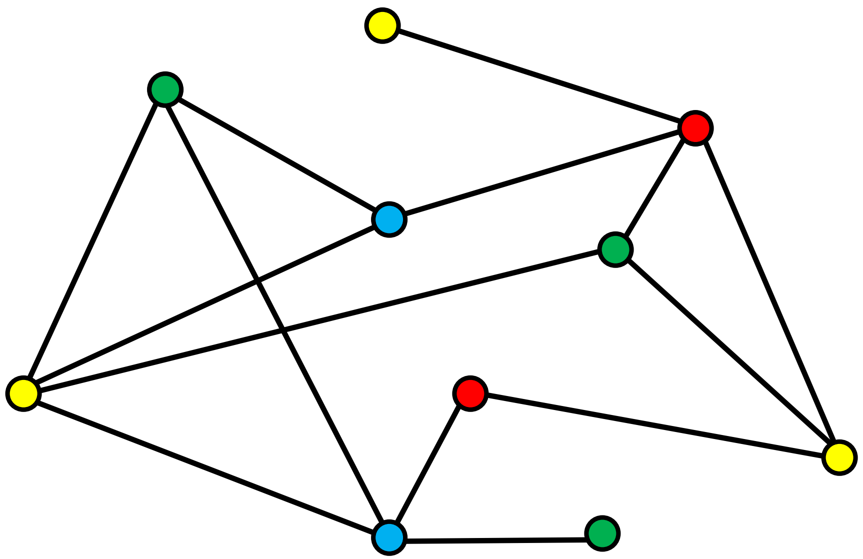
Complete graphs



K_1 K_2 K_3 K_4 K_5

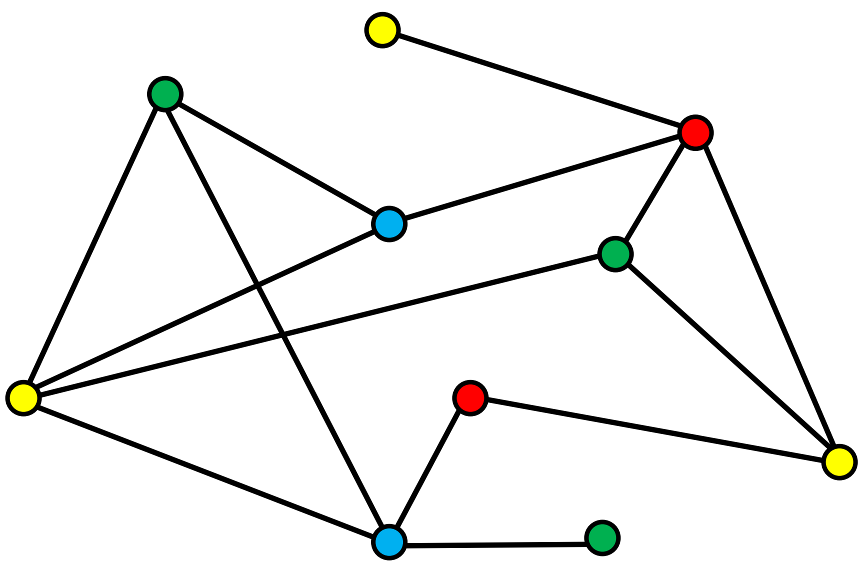
Coloring a graph

Coloring a graph means coloring its vertices so that adjacent vertices get different colors



Coloring a graph

$\chi(G)$: smallest possible number of colors
with which a graph G can be colored



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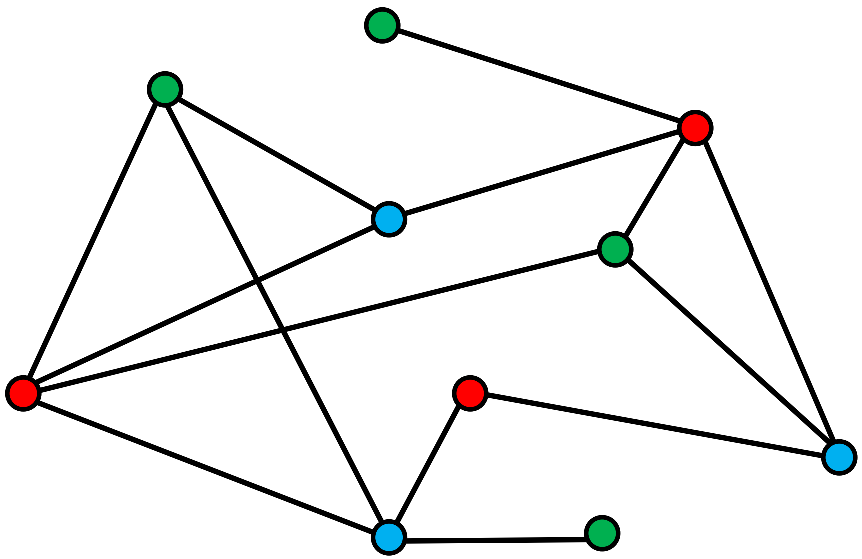
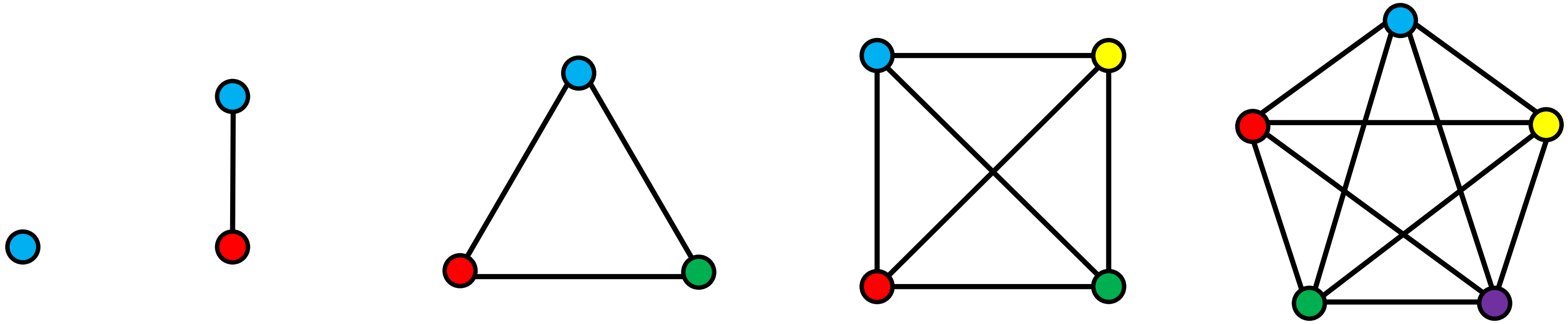


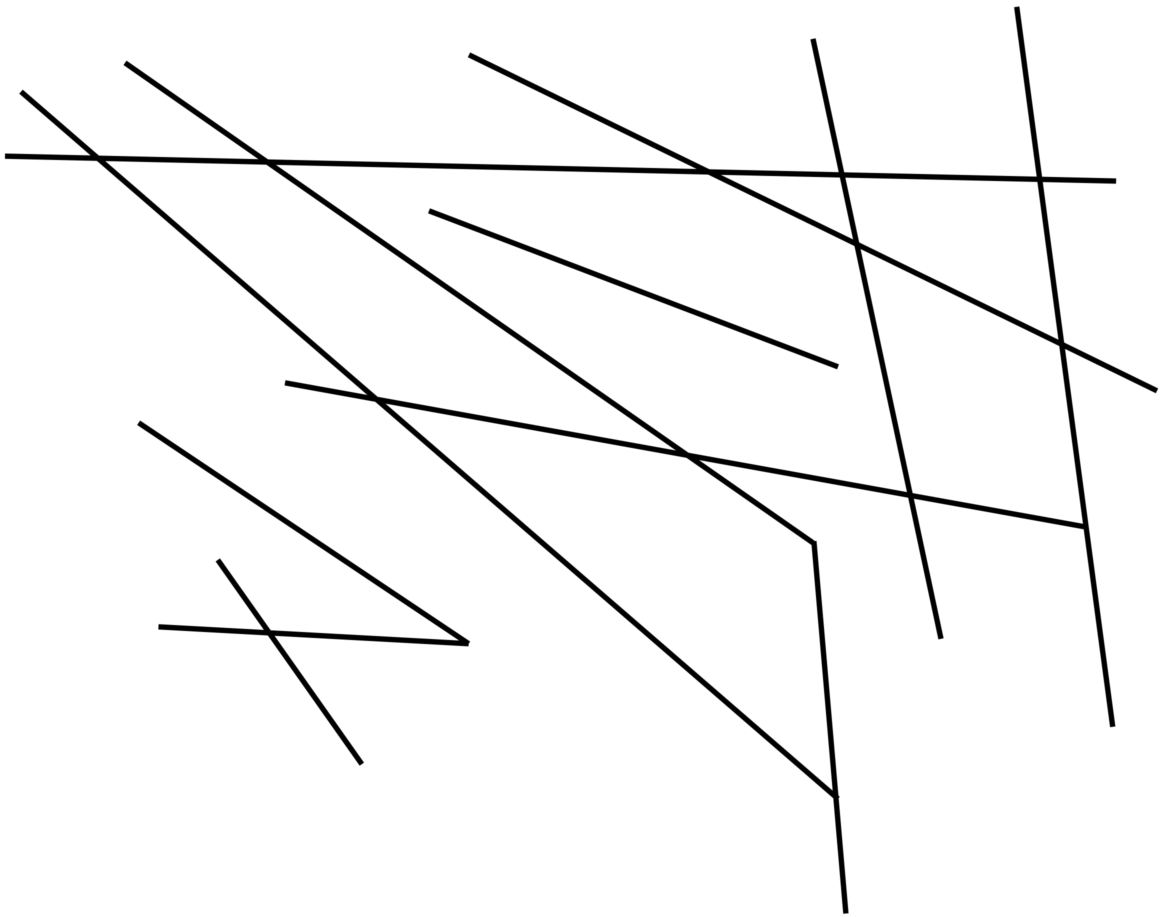
$\chi(G) = 3$

Coloring a graph



$$\chi(K_n) = n$$

Set of line segments



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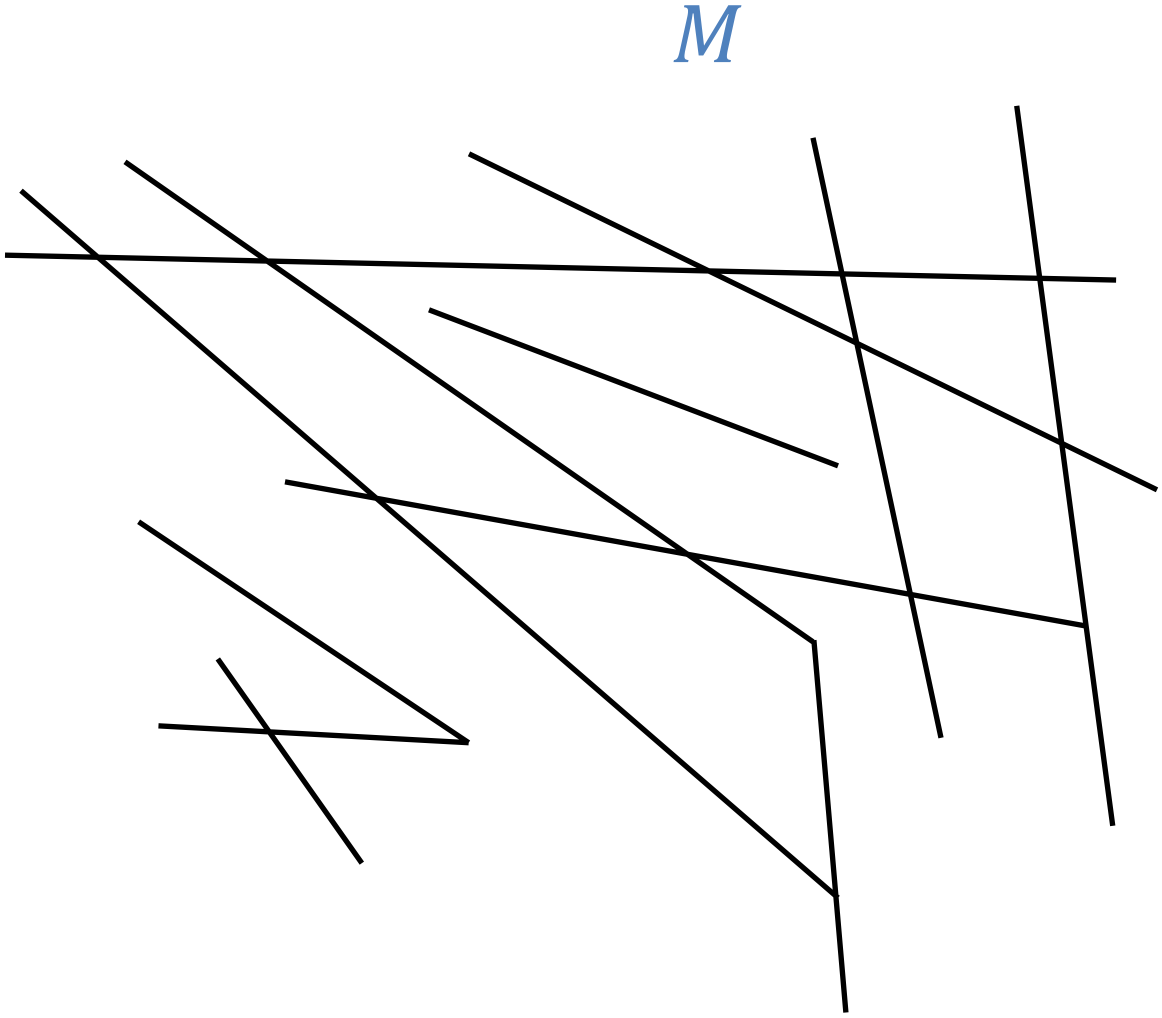
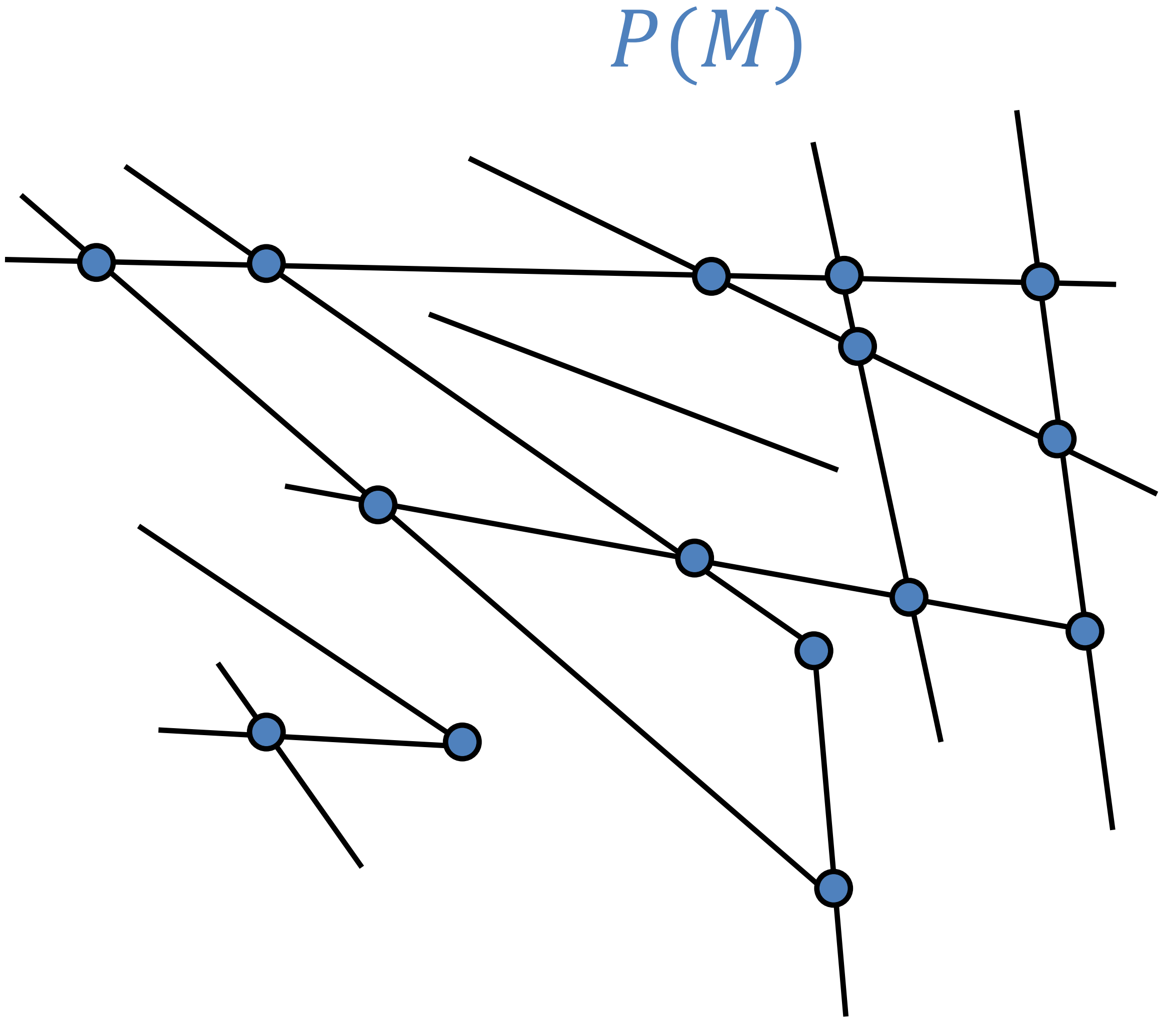
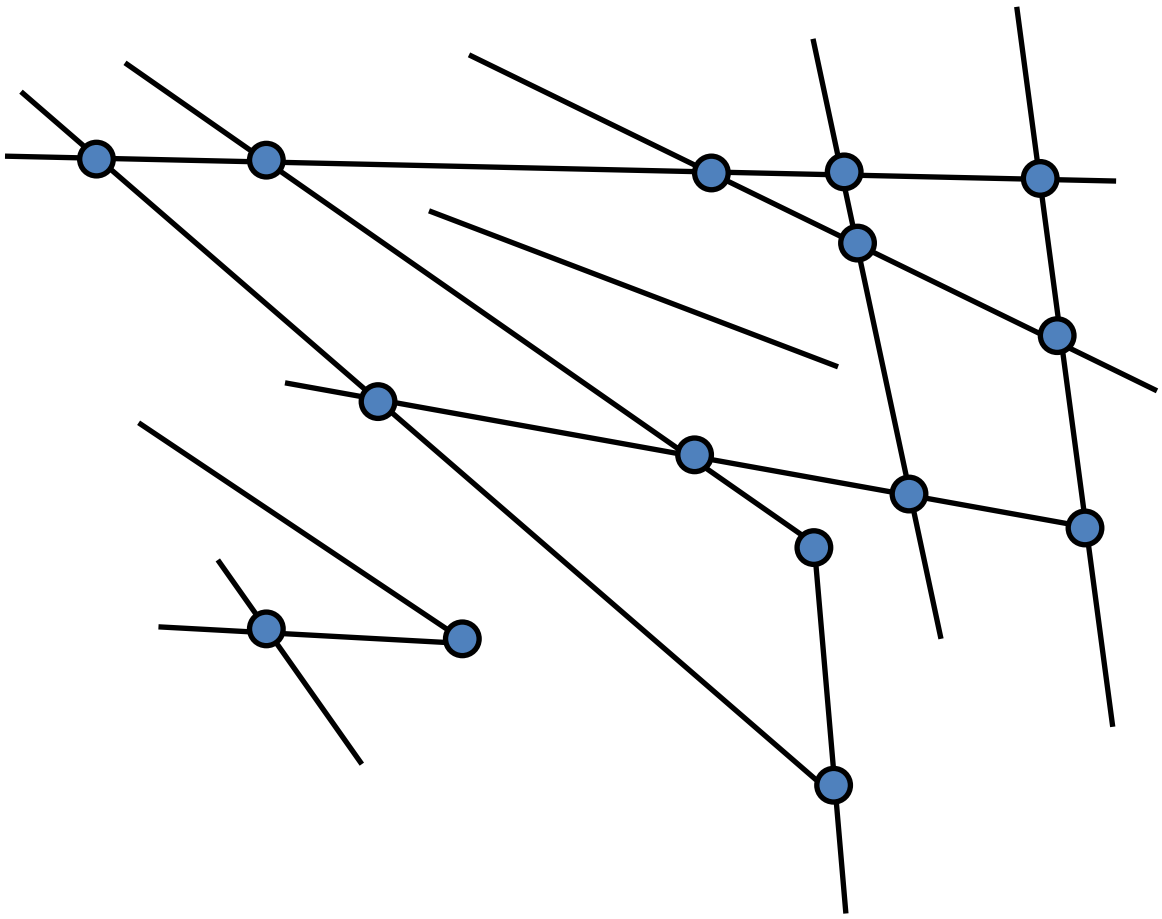


Intersection points



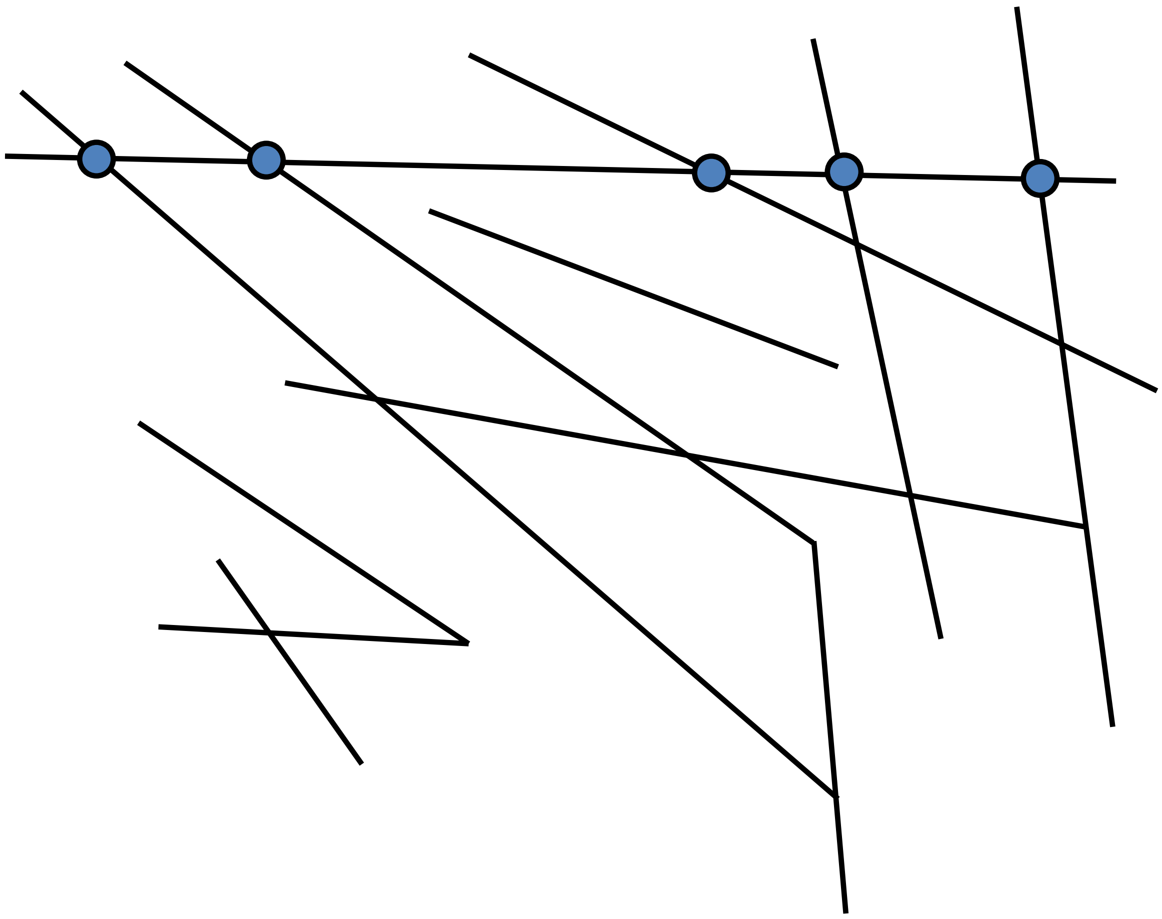
Intersection points

$w(M)$: most intersection points in a segment

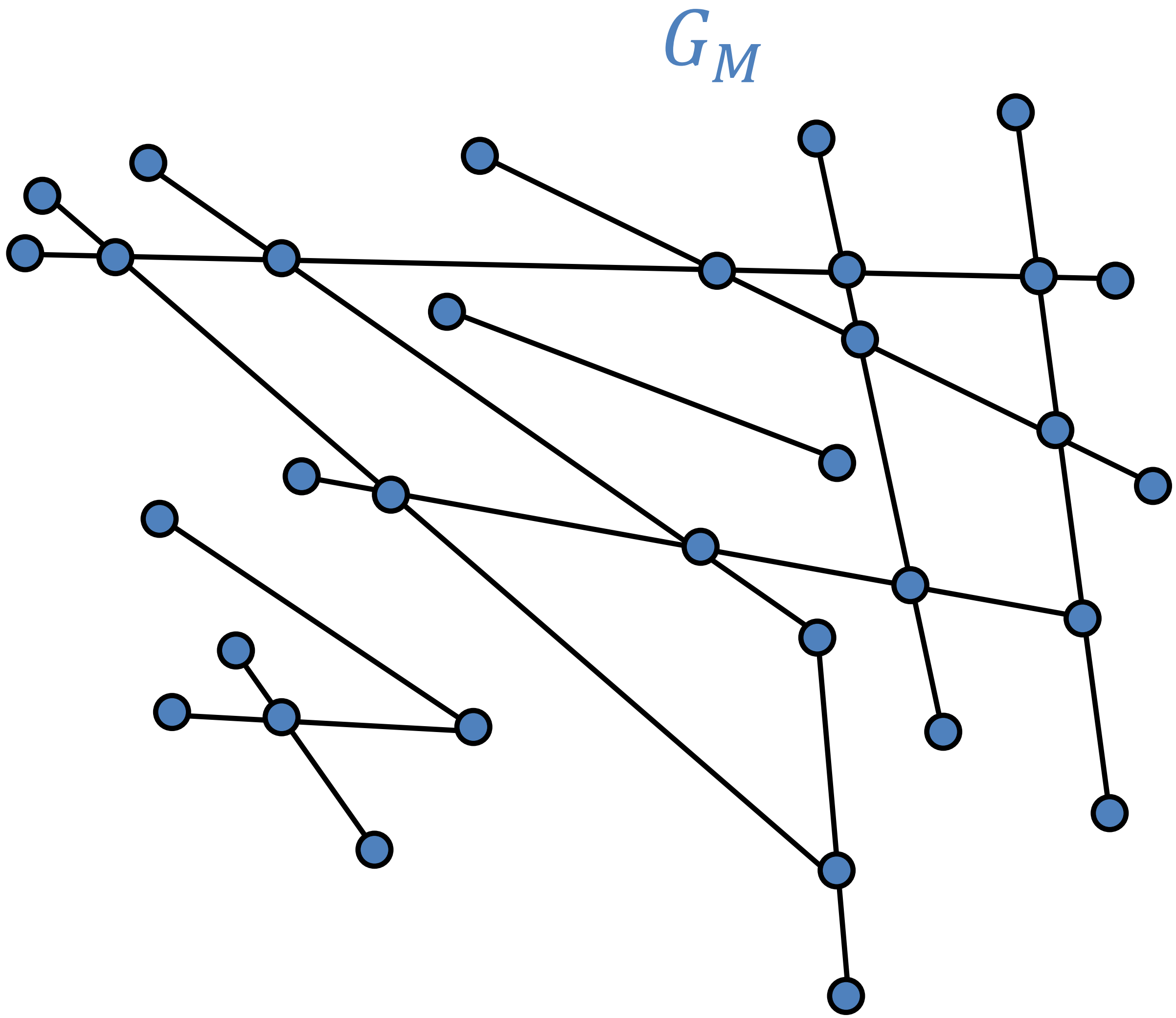


Intersection points

$w(M) = 5$

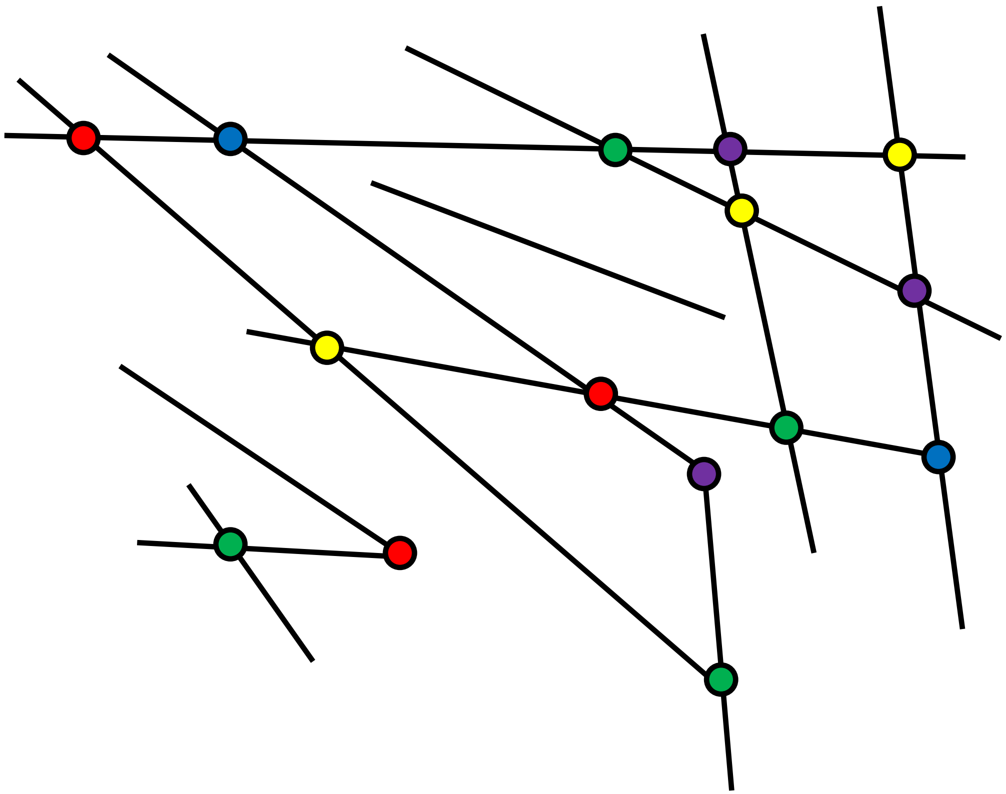


Corresponding graph



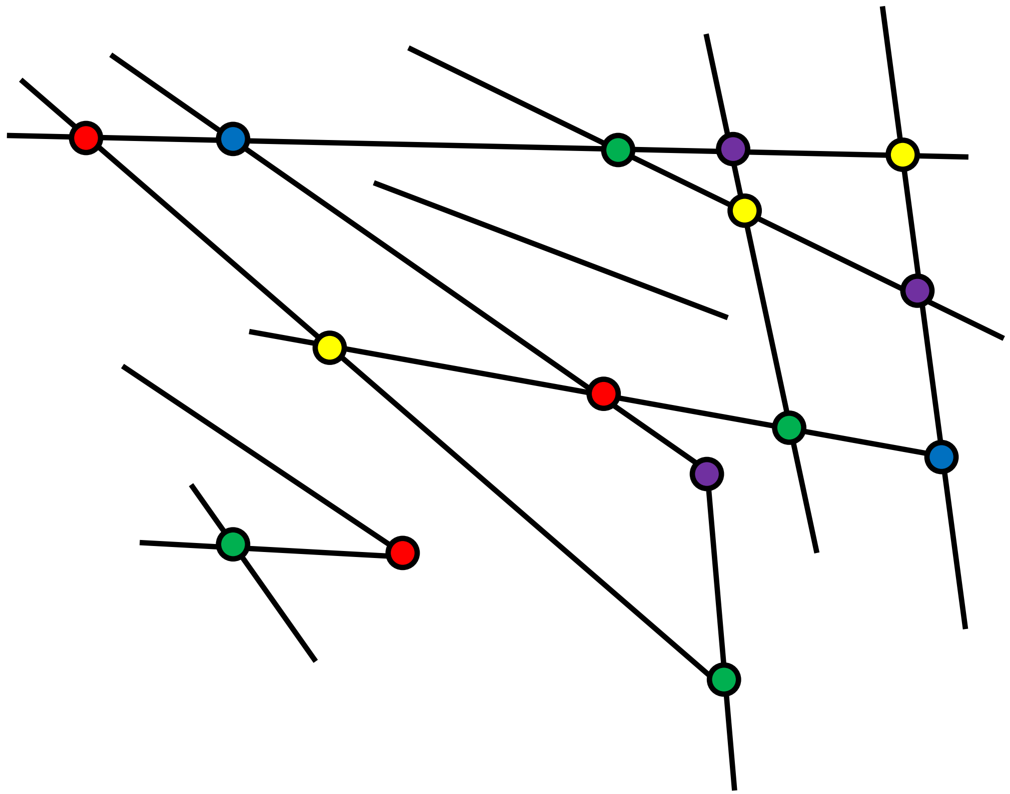
Coloring a set of segments

Coloring a set of segments means coloring the intersection points so that no segment has points with the same color.



Coloring a set of segments

$\chi(M)$: smallest possible number of colors with which a set of segments M can be colored



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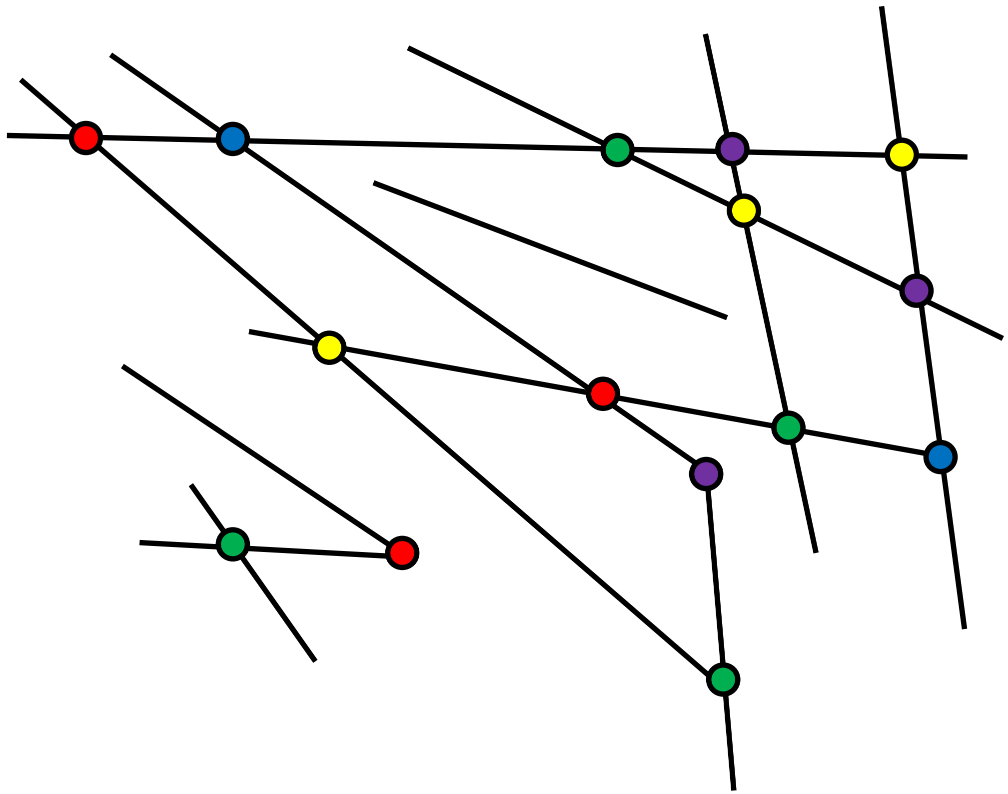

$\chi(M) = 5$

Erdős-Faber-Lovász Conjecture

Let G be a graph consisting of n copies of K_n, every pair of which has at most one vertex in common. Then, $\chi(G) = n$.

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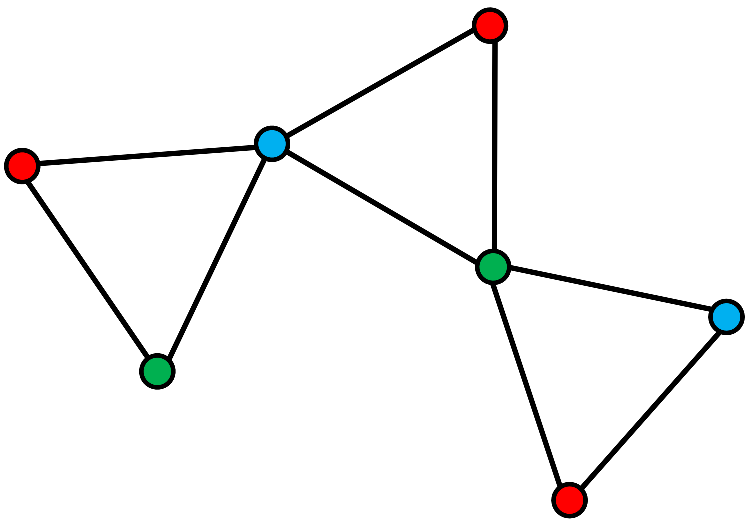

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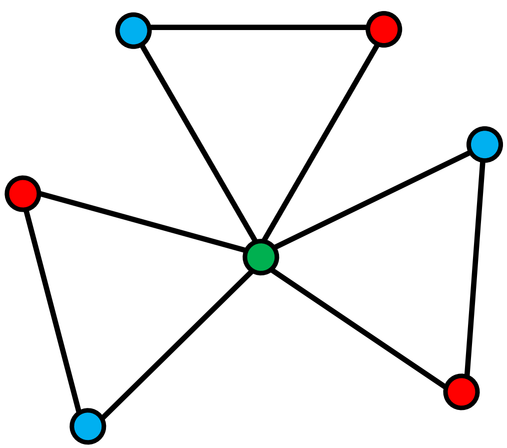

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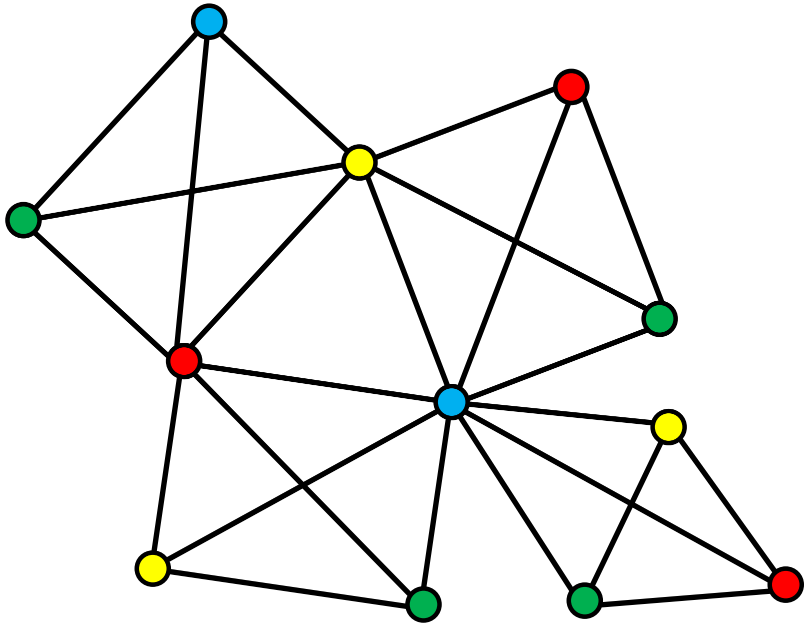

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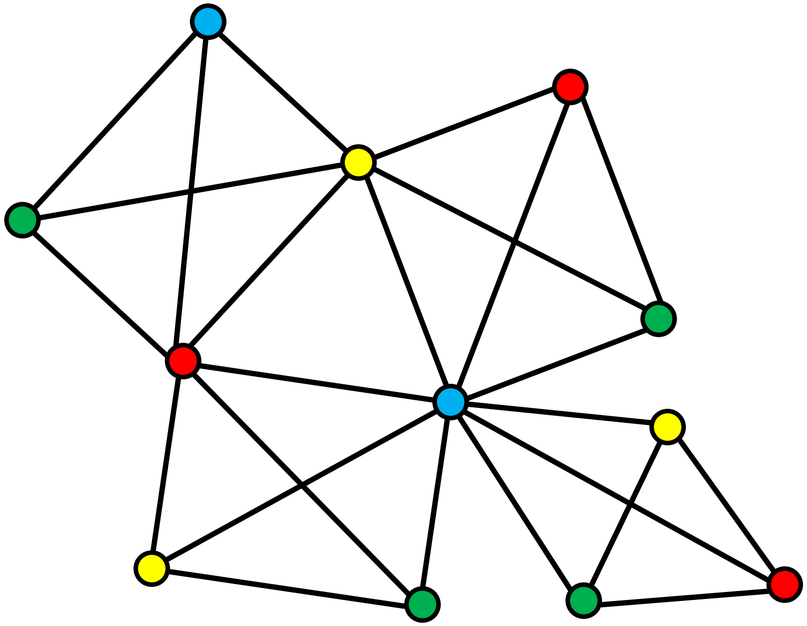

$\chi(G)$ cannot be less than n, since at least n colors are needed for each copy of K_n

Erdős-Faber-Lovász Conjecture

Let G be a graph consisting of n copies of K_n, every pair of which has at most one vertex in common. Then, $\chi(G) = n$.

[Kang et al. 2021] announced a proof for large enough n

Erdős-Faber-Lovász Conjecture

Let M be a set of m curves, each pair of which has at most one point in common. Then, $\chi(M) \leq m$.

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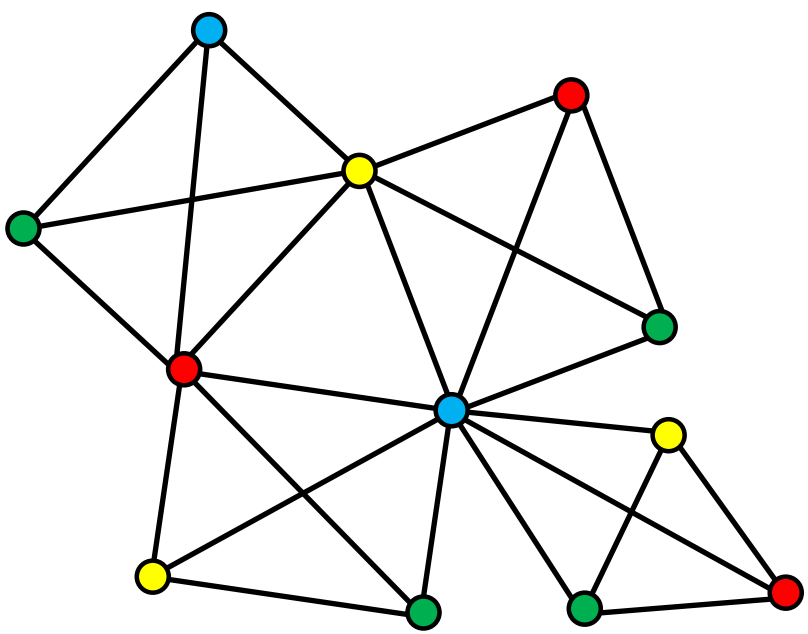

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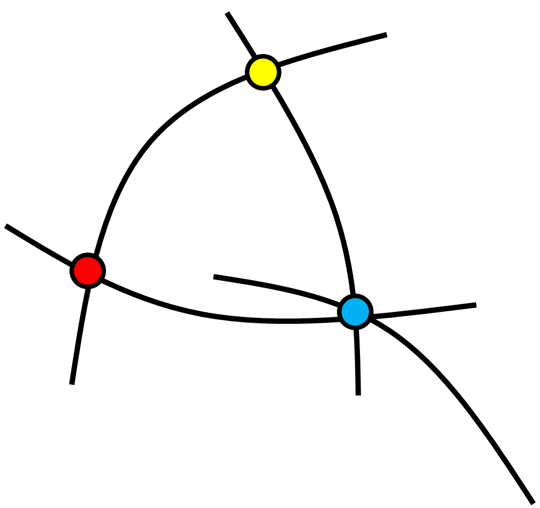

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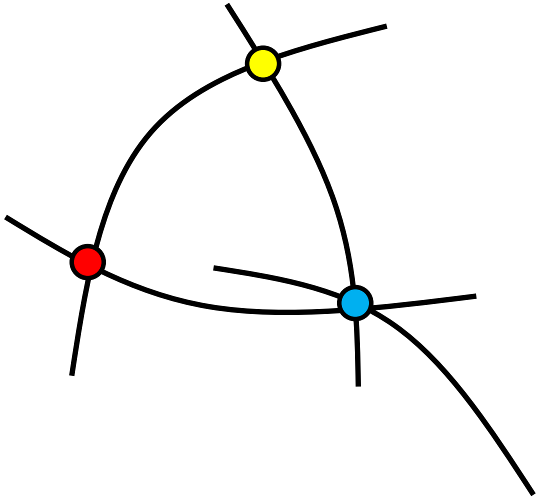
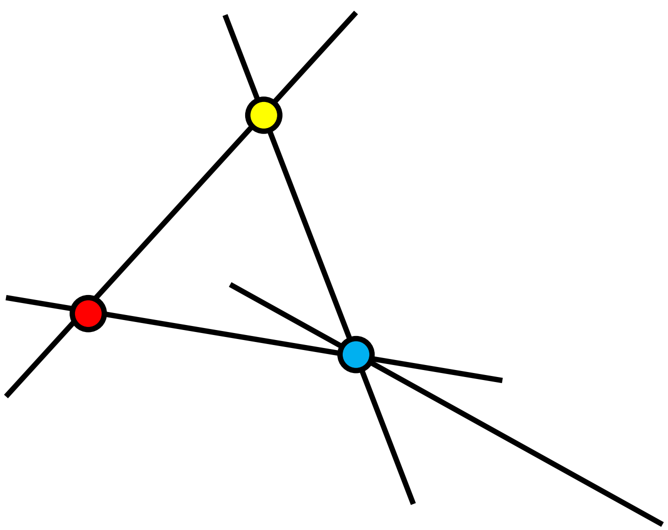


$\chi(M)$ can be less than m.

Coloring a set of segments

Let M be a set of m segments. What's $\chi(M)$?



Bounds

$$w(M) \leq \chi(M) \leq |M|$$

Bounds

Need $w(M)$ colors for the segment with $w(M)$ intersections

Follows from proof of EFL Conjecture

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Bounds

Need $w(M)$ colors for the segment with $w(M)$ intersections

Follows from proof of EFL Conjecture

$$w(M) \leq \chi(M) \leq |M|$$

easy to compute

Complexity of $\chi(M)$

Computing $\chi(M)$ is NP-Complete, even if M is a set of segments where no five segments intersect in the same point.

Reduction using a special case of PLANAR GRAPH COLORING

Bounds

Need $w(M)$ colors for the segment with $w(M)$ intersections

Follows from proof of EFL Conjecture

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Need $w(M)$ colors for the segment with $w(M)$ intersections

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Questions

Experimentally, $\chi(M)$ usually seems to be a lot closer to $w(M)$ than to $|M|$.

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- Can $\chi(M) \geq w(M) + 3$?

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- When is $\chi(M) = w(M)$?
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- When is $\chi(M) = w(M) + 2$?

} Characterizations are hard to derive

- Can $\chi(M) \geq w(M) + 3$?

} Hard to find one

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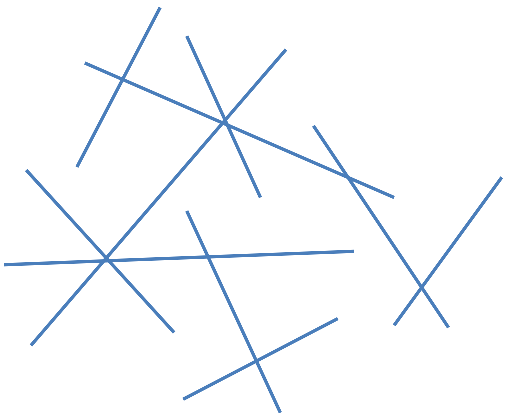
- Can $\chi(M) \geq w(M) + 3$?

What is $\chi(M)$ for M with special structure?

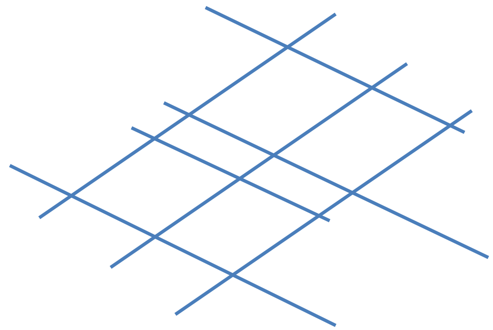
Easier to answer

Some characterizations



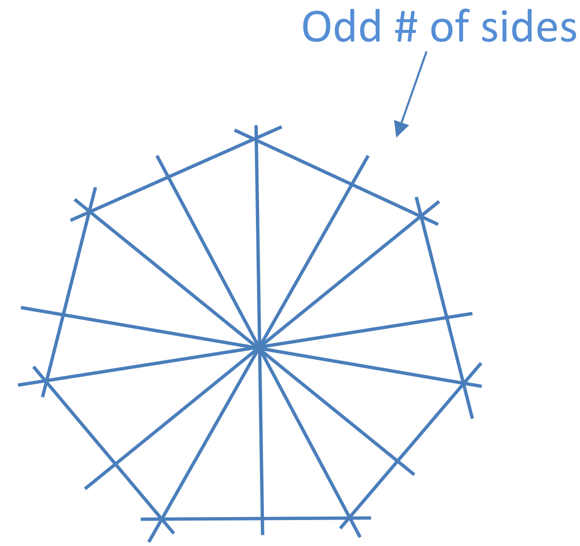
"circuit-free arrangements"

$\chi(M) = w(M)$



"grid-like arrangements"

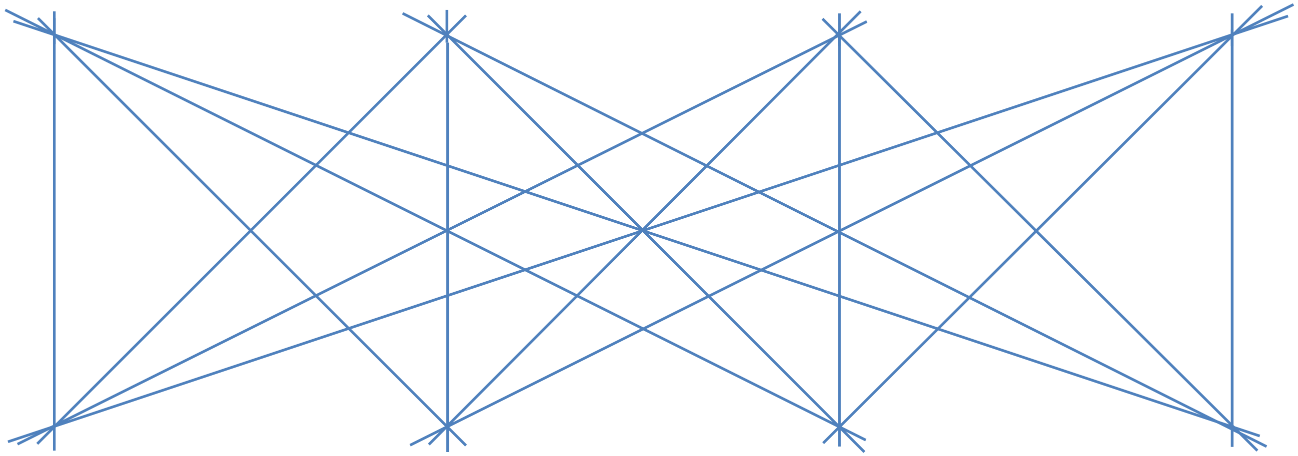
$\chi(M) = w(M)$



"bisected polygon arrangements"

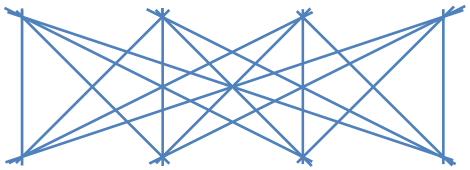
$\chi(M) = w(M) + 2$

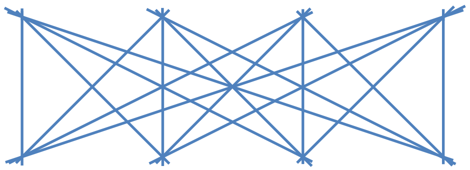
Some characterizations



"standard drawing of $K_{n,n}$"

$$\chi(M) = |\{(x, y): 1 \leq x, y \leq n, \gcd(x, y) = 1\}| + 2$$

Approach for  family

Approach for  family

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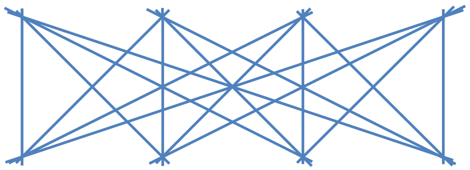
Approach for  family

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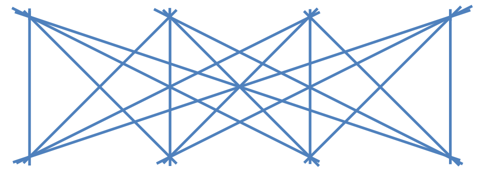
Approach for  family

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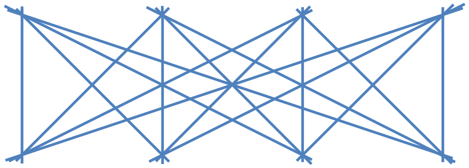
Approach for  family

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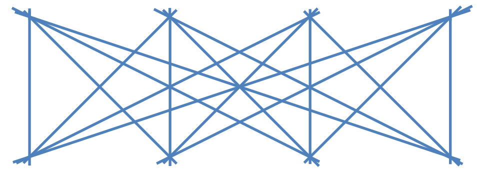
Approach for  family

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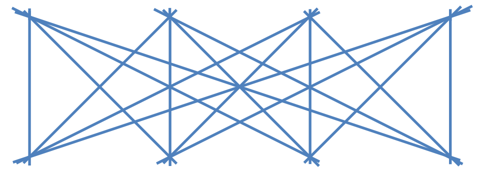
Approach for  family

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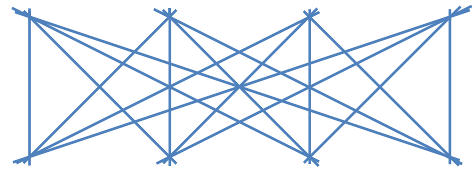
Approach for  family

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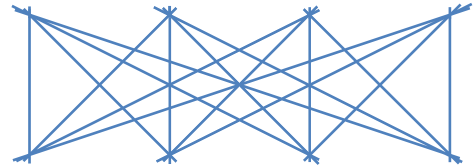
Approach for  family

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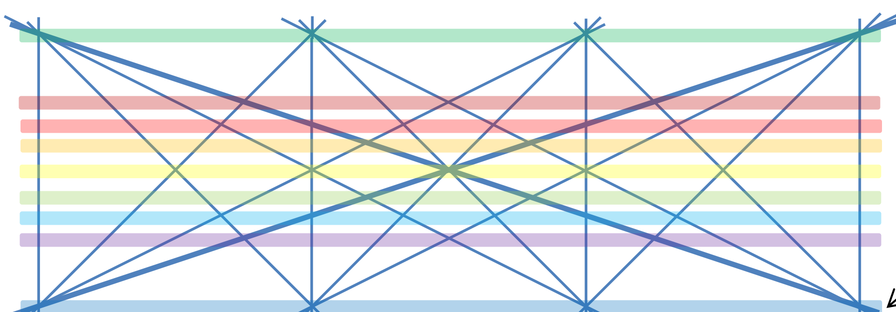
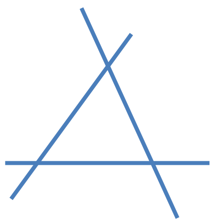
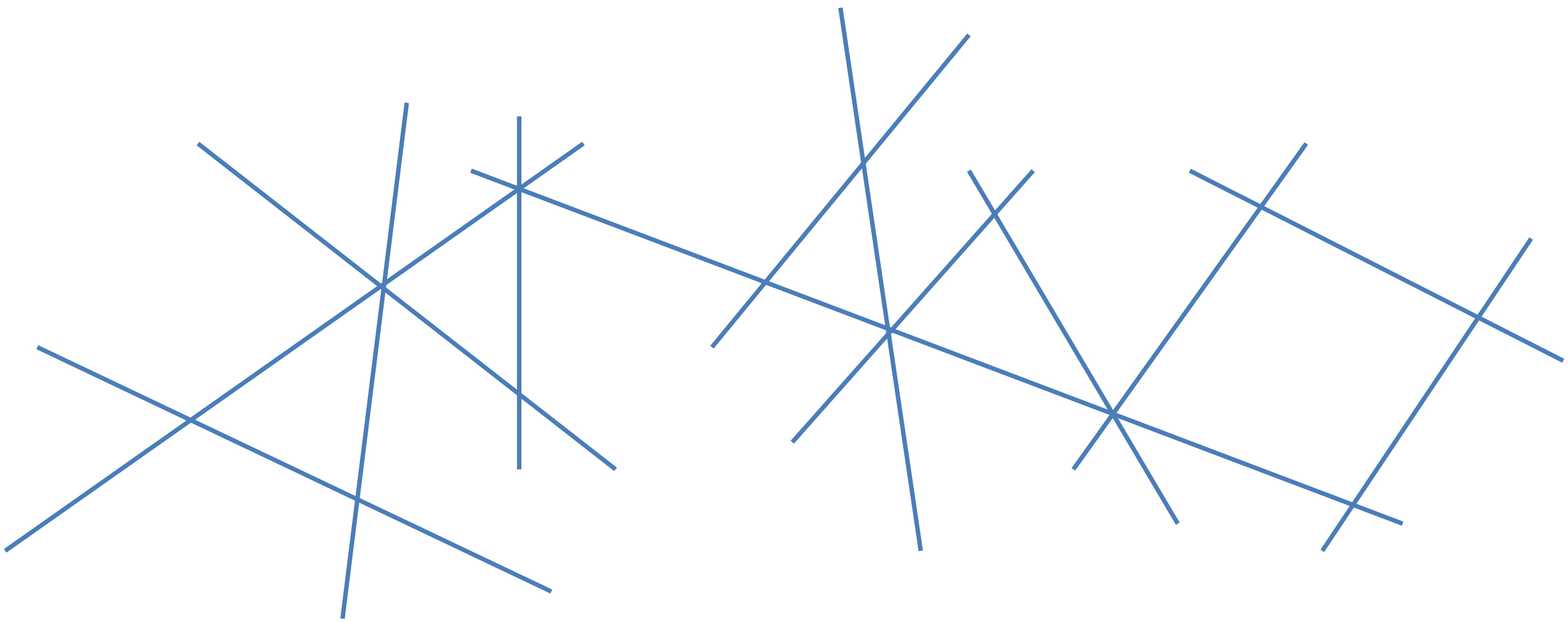


Longest diagonal segments intersect all others, so $w(M)$ colors are necessary, and also sufficient

Some characterizations

If M is a segment cactus different from  ,
then $\chi(M) \leq |M| - 1$.



What about lines?

In a set M of lines, every pair of non-parallel lines intersect

$w(M) \leq \chi(M) \leq |M|$ still holds

What about lines?

The following are equivalent:

- If M is a set of m lines drawn in the plane, then M has a coloring with m colors.
- If M is a set of m segments drawn in the plane, then M has a coloring with m colors.

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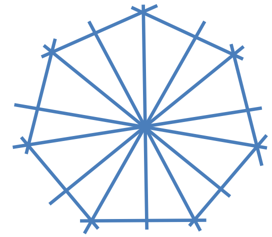
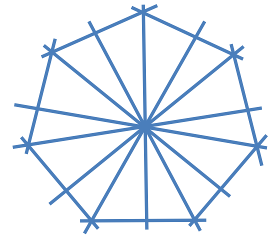
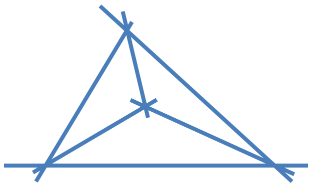
An analogous statement does not necessarily hold for subfamilies of segments/lines

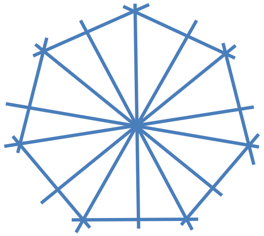
Approach for  family

Approach for  family

- Randomly noticed one small example



$$\chi(M) = w(M) + 2$$

Approach for  family

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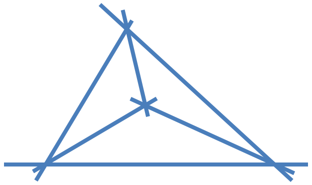
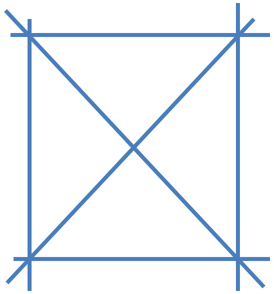


$$\chi(M) = w(M) + 2$$

- Noticed another that was similar to it



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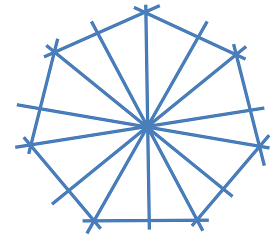
Approach for  family

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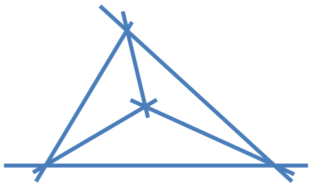

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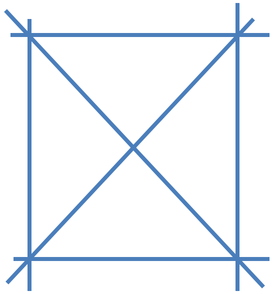
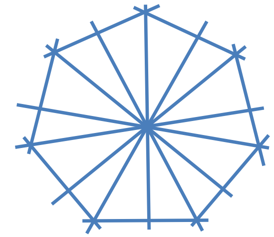
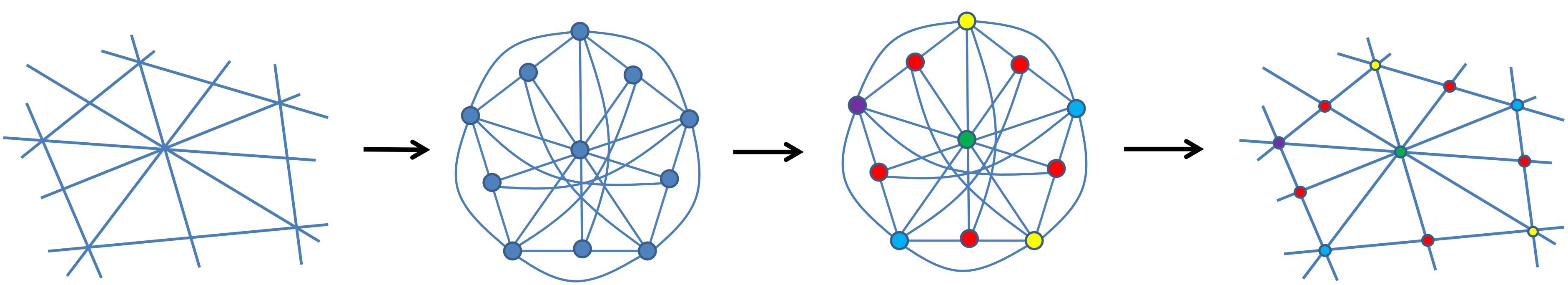


$$\chi(M) = w(M) + 2$$

- Tried to extend the pattern
- Harder to verify large examples by hand

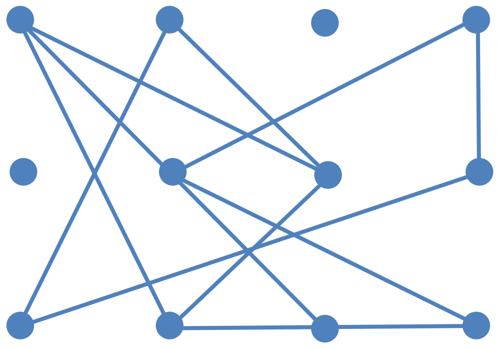
Approach for  family

- Wrote integer program to help verification
- Used program to check more examples
- Saw $\chi(M) = w(M) + 2$ holds for all of them
- Analytically proved formula
  - Proof is based on converting the set of segments to a graph, finding a coloring that always works, and showing that no smaller coloring exists



Future work

- Find a set with $\chi(M) \geq w(M) + 3$ or show this can't exist.
- Run more extensive computational experiments



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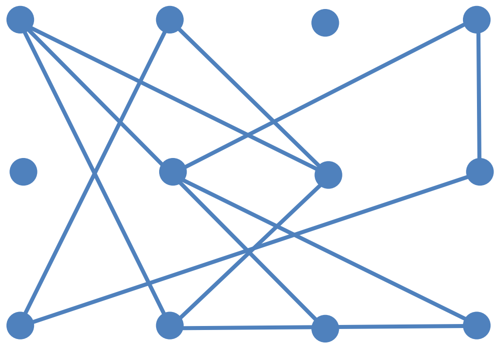

Thank you